

Electrostatic self-energy in QED_2 on curved background

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Abstract

By considering the vacuum polarization, we study the effects of geometry on electrostatic self-energy of a test charge near the black hole horizon and also in regions with strong and weak curvature in static two dimensional curved backgrounds. We discuss the relation of ultraviolet behavior of the gauge field propagator and charge confinement.

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1 Introduction

One of the interesting areas in physics, is the study of behavior of classical and quantum fields in curved background and investigating how their properties are affected by the curvature of the space-time. It also provides a lot of insights into important problems such as black hole entropy, Hawking radiation, quantum theory of gravity and so on.

It is well known that the change in the geometry of the space associated with the gravitational field, deforms electromagnetic field, inducing a self-force on a point charge at rest in a static curved space-time[1].

Also the presence of boundary condition like the boundary condition at conical singularity produced for example by a cosmic string or a point mass, alters the electromagnetic field of a point charge which after subtracting the infinite part, leads to a finite self-force [2].

A renewed investigation has been appeared in this subject in order to study the upper bound on the entropy of charged object by requiring the validity of thermodynamics of black-holes. This problem is studied in [3], for classical black-hole backgrounds, in the absence of dynamical fermions, i.e. disregarding vacuum polarization.

Another subject of studies in gauge field theory is the screening and confinement of charges. The static potential between external charges, which can be obtained from the Wilson loop expectation value, carries important information of infrared behavior of gauge fields which is suggested to be responsible for confinement, binding the quarks and anti-quarks into $q\bar{q}$ pairs (infrared slavery). Because of computational hurdles in four dimensions one can consider these problems in lower dimensional models, as a laboratory to study physical effects which can be carried out to the real world.

In this letter we study the influence of curvature on the self-energy of static charges, by considering the effect of vacuum polarization in two dimensional static space-times. We show that in order to explain the confining behavior of QED_2 [5] on a curved background, considering ultraviolet behavior of gauge fields and self-energy of external charges is necessary, in other words our method of studying confinement involves the behavior of two point function in the ultraviolet regime instead of the infrared. To do so We use the criterion expressing that in the confinement phase the energy of an isolated quark is infinite. In usual calculations in four dimensional flat space-time the self-energy of a test charge is infinite and is subtracted from the potential energy, but on curved space-times the finite part of self-energy is not a constant and must be considered in computing the forces[2]. In two dimensions this self-energy is related to the Green function of a Sturm- Liouville type operator at coincident limit and therefore is analytic. We show that, for QED_2 , the self-force can prohibit a single charge to be in some region of the space unless it is coupled to another opposite test charge, forming mesonic $q\bar{q}$ structure. We also obtain electrostatic self-force using the heat kernel method up to the adiabatic order four.

2 Geometrical effects on charge confinement and mesonic structure

A general static two dimensional surface can be described by the metric

$$ds^2 = \sqrt{g}(x)(dt^2 - dx^2), \quad (1)$$

where $\sqrt{g}(x)$ is the conformal factor.² On this space-time, QED_2 consisting of charged matter field interacting with an abelian gauge field in two dimensions is described by Lagrangian

$$L = \sqrt{g}(x)(\psi^\dagger \gamma^\mu (\nabla_\mu - ieA_\mu) \psi) + \frac{1}{2\sqrt{g}(x)} F^2, \quad (2)$$

where γ^μ are the curved space counterparts of Dirac gamma matrices. ∇_μ is the covariant derivative, including the spin connection, acting on fermionic fields. e is the charge of dynamical fermions. The dual field strength F , is described through $F = \epsilon^{\mu\nu} \partial_\mu A_\nu$, where $\epsilon^{\mu\nu} = \epsilon_{\mu\nu}$ and $\epsilon^{01} = -\epsilon_{10} = 1$. By integrating out matter fields one can obtain one loop effective action for the gauge field [4]

$$L_{eff.} = \frac{1}{2\sqrt{g}(x)} F^2 + \frac{\mu^2}{2} F \frac{1}{\partial^2} F, \quad (3)$$

where $\mu = \frac{e}{\sqrt{\pi}}$. In static case and using the Coulomb gauge $A_1 = 0$ this Lagrangian reduces to

$$L_{eff.} = \frac{1}{2\sqrt{g}(x)} \left(\frac{dA_0}{dx} \right)^2 + \frac{\mu^2}{2} A_0^2. \quad (4)$$

Hence in the presence of vacuum polarization, the gauge field has gained a mass via a peculiar two dimensional version of Higgs phenomenon. As a consequence, one may expect the replacement of the Coulomb force by a finite range force. We introduce two static opposite charges located at $x = a$ and $x = b$, described by the covariantly conserved current

$$J^0(x) = \frac{e'}{\sqrt{g}(x)} (\delta(x - b) - \delta(x - a)), \quad J^1 = 0. \quad (5)$$

The gauge field's equation of motion is

$$\frac{d}{dx} \frac{1}{\sqrt{g}(x)} \frac{dA_0}{dx} - \mu^2 A_0 = e' (\delta(x - b) - \delta(x - a)). \quad (6)$$

The Green function of the elliptic operator $\frac{d}{dx} \frac{1}{\sqrt{g}(x)} \frac{d}{dx} - \mu^2$ satisfies

$$\left(\frac{d}{dx} \frac{1}{\sqrt{g}(x)} \frac{d}{dx} - \mu^2 \right) G(x, x') = \delta(x, x'). \quad (7)$$

²Euclidean version of this space-time is $ds^2 = \sqrt{g}(x)(dt^2 + dx^2)$, whose one of the geodesics is the straight line parallel to the x axis.

In terms of $G(x, x')$ the energy of external charges is obtained

$$E = \int T_0^0 dx = - \int L_{eff} dx = -\frac{e'^2}{2}[G(a, a) + G(b, b) - 2G(a, b)]. \quad (8)$$

This is the energy measured by an observer whose velocity $u^\mu = (g^{-1/4}(x), 0)$ is parallel to the direction of the time-like killing vector of the space-time and $-\frac{e'^2}{2}G(x, x)$ is the self-energy of a static point charge located at x .

On flat surfaces this change of energy is

$$E = \frac{e'^2}{2\mu}(1 - e^{-\mu|b-a|}). \quad (9)$$

This can be obtained by using the gauge fields Green's function

$$G(x, x') = -\frac{1}{2\mu}e^{-\mu|x-x'|} \quad (10)$$

Note that $-\frac{e'^2}{2}G(x, x)$, in contrast to higher dimensions has a finite value. This is due to finiteness of Helmholtz Green's function in coincident limit in one dimension. For distant charges as a result of screening, interaction term becomes zero, and E tends to self-energy of test charges, which in flat case is $E_{|b-a|\rightarrow\infty} = \frac{e'^2}{2\mu}$ [6]. Note that although this self-energy is finite, but it is a constant therefore the self-force is zero. As we will show on a curved surface, the Green function of the Sturm-Liouville operator (7) is an analytic function at coincident limit and self-force becomes position dependent.

Besides, the role of dynamical charge is affected by the presence of the curvature

$$R(x) = \frac{1}{\sqrt{g}(x)} \frac{d}{dx} \frac{1}{\sqrt{g}(x)} \frac{d}{dx} \sqrt{g}(x), \quad (11)$$

To see this we write the equation (7) as

$$\frac{1}{\sqrt{g}(x)} \frac{d}{dx} \frac{1}{\sqrt{g}(x)} \frac{d}{dx} \sqrt{g}(x) \tilde{G}(x, x') - \mu^2 \tilde{G}(x, x') = \frac{\delta(x, x')}{\sqrt{g}(x)}, \quad (12)$$

where $G(x, x') = \sqrt{g}(x) \tilde{G}(x, x')$. This equation can be rewritten as

$$(R(x) - \mu^2) \tilde{G}(x, x') + \frac{1}{g(x)} \left(\frac{d\sqrt{g}(x)}{dx} \right) \left(\frac{d\tilde{G}(x, x')}{dx} \right) + \frac{1}{\sqrt{g}(x)} \frac{d^2 \tilde{G}(x, x')}{dx^2} = \frac{\delta(x, x')}{\sqrt{g}(x)}, \quad (13)$$

which is equivalent to

$$(R(x) - \mu^2) G(x, x') + \frac{1}{\sqrt{g}(x)} \left(\frac{d\sqrt{g}(x)}{dx} \right) \left(\frac{d}{dx} \frac{1}{\sqrt{g}(x)} G(x, x') \right) + \frac{d^2}{dx^2} \left(\frac{G(x, x')}{\sqrt{g}(x)} \right) = \delta(x, x'). \quad (14)$$

Hence in the strong curvature limit $|R(x)| \gg \mu^2$, the Green function is approximately unaffected by dynamical fermions. In other words if we assume the same boundary condition for

the gauge field in the presence and absence of dynamical fermions, the vacuum polarization, in contrast to the flat case, doesn't change the energy and confining phase of the system.

To elucidate this subject and to emphasize how the ultraviolet behavior of QED_2 on curved space-time is concerned in confinement of test charges let us give an example. Consider the following space-time in conformal coordinates

$$ds^2 = \frac{dt^2 - dx^2}{x^m}; \quad x > 0 \quad (15)$$

where $m = 2 - 1/(k+1)$; $k \neq 0, -1/2$, this is one of the classical solutions of two dimensional scale invariant gravity [7] with a curvature singularity. Here we assume that this is only a classical background for QED_2 . The homogenous solutions $G_h(x)$ of the equation (7) satisfy

$$\frac{d}{dx} x^m \frac{d}{dx} G_h(x) - \mu^2 G_h(x) = 0. \quad (16)$$

For $m \neq 2$, by defining $z \equiv x^{1-\frac{m}{2}}$ and $G_h(x) \equiv x^{\frac{1-m}{2}} u$, we obtain

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - \left(\frac{(m-1)^2}{(m-2)^2} + \frac{4\mu^2 z^2}{(m-2)^2} \right) u = 0. \quad (17)$$

Hence

$$G_h(x) = x^{\frac{1-m}{2}} \begin{cases} I_{|\frac{m-1}{m-2}|} \left(\sqrt{\frac{4\mu^2}{(2-m)^2}} x^{1-\frac{m}{2}} \right) \\ K_{|\frac{m-1}{m-2}|} \left(\sqrt{\frac{4\mu^2}{(2-m)^2}} x^{1-\frac{m}{2}} \right). \end{cases} \quad (18)$$

Therefore the Green function is

$$G(x, x') = -\frac{2}{|m-2|} (x_{<} x_{>})^{\frac{1-m}{2}} \begin{cases} I_{\frac{m-1}{m-2}} \left(\frac{2\mu}{m-2} x_{>}^{1-\frac{m}{2}} \right) K_{\frac{m-1}{m-2}} \left(\frac{2\mu}{m-2} x_{<}^{1-\frac{m}{2}} \right) & \text{if } m > 2 \\ I_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{2-m} x_{<}^{1-\frac{m}{2}} \right) K_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{2-m} x_{>}^{1-\frac{m}{2}} \right) & \text{if } m < 2, \end{cases} \quad (19)$$

where $x_{>(<)}$ is the bigger (smaller) of x, x' and I, K are modified Bessel functions. This Green function Satisfies Dirichlet boundary condition at $x = 0$ and at $x = \infty$. At the coincidence limit the Green function is

$$G(x, x) = -\frac{2}{|m-2|} x^{1-m} I_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{|m-2|} x^{1-\frac{m}{2}} \right) K_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{|m-2|} x^{1-\frac{m}{2}} \right). \quad (20)$$

In terms of the scalar curvature $R(x) = \frac{m}{x^{2-m}}$, this relation becomes

$$G(x, x) = -\frac{2}{|m-2|} \left(\frac{m}{R(x)} \right)^{\frac{1-m}{2-m}} I_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{|m-2|} \left(\frac{R(x)}{m} \right)^{-\frac{1}{2}} \right) K_{|\frac{m-1}{m-2}|} \left(\frac{2\mu}{|m-2|} \left(\frac{R(x)}{m} \right)^{-\frac{1}{2}} \right). \quad (21)$$

In the strong curvature limit, or in regions where $(|R(x)| \gg \mu^2)$, by considering the asymptotic behavior of Bessel functions this relation becomes

$$\lim_{R \rightarrow \infty} G(x, x) = -\left| \frac{k+1}{k} \right| \left(\frac{R(x)}{m} \right)^k, \quad (22)$$

which is μ independent as anticipated. k is defined after the equation (15). In regions where the curvature is weak $|R(x)| \ll \mu^2$, we have

$$G(x, x) = -\frac{1}{2\mu} \left(\frac{R(x)}{m} \right)^{\frac{m}{2(2-m)}}. \quad (23)$$

Note that for $m = 0$, $G(x, x) = -\frac{1}{2\mu}$, which is the same as the result on a flat uncurved surface [6].

Now we show the relation of electrostatic self-energy and confinement. $G(x, x)$ in (21) is an analytic non-constant function for $x > 0$ (which may become very large in some regions, signaling, as we will show, a confining situation), and therefore we expect that the charge feels an electrostatic self-force, affected by the geometry of the space-time as well as the boundary condition imposed on the gauge fields.

We use the fact that in the confinement phase the energy of an isolated quark is infinite.

Assume that $m > 2$; the energy needed to locate a single charge e' in the region ($R(x) \neq 0$) $\ll \mu^2$ or $x \simeq 0$ (in the coordinate (15)), is

$$E_{self.}(x) = \frac{e'^2}{4\mu} \left(\frac{R(x)}{m} \right)^{\frac{m}{2(2-m)}}, \quad (24)$$

which tends to infinity. In other words there is a great repulsive force on an external charge near $x = 0$ prohibiting to have single charges in this region, or the energy of an isolated charge in this region is very large. The same procedure occurs in the region $x \simeq 0$, for $m < 2$ (when $|R(x)| \rightarrow \infty$). So in these regions following the equation (8) only charges forming mesonic structure may survive. These two opposite charges must be near together in order to obtain a finite energy for the system. Our criterion for confinement is not based on the behavior of the energy in the infrared (where the geodesic distance of external charges tends to infinity).

In QED_4 the dielectric constant of vacuum is larger than unity as a result of the screening effect due to vacuum polarization. If instead one consider a case in which the dielectric constant of the vacuum vanishes then due to antiscreening effect the energy of the system becomes infinite unless we add another opposite test charge to the system then this fictitious system is confining. A similar phenomenon occurs in dual bag model. In that case quarks and anti-quarks must form mesonic structure in order to avoid divergences in static potential, i.e only color singlets have finite energy because the divergence term appears as a multiplier of total external charges [8]. In our model the role of dielectric constant in confining phase is played by the metric components (see equation (2)). To find some relations between geometry and permittivity see [9].

The repulsive forces on single charges besides the curvature of the space-time is related to the boundary conditions imposed on the gauge fields in defining the vacuum of the system,

for example in equation (26), although R is small but we have a great repulsive force. These repulsive forces may also be arisen on a surface with constant curvature (adS or dS space-times, obtained for example by taking $m = 2$ in the previous example)[10]. Also Maxwell field theory (disregarding vacuum polarization) in 2+1 dimensional conical space-times, despite the null curvature of the manifold exhibits a repulsive force on charges [2].

In the previous example we considered a space-time with a naked singularity, in this part we study the electrostatic self-energy on a black hole background.

For Maxwell theory on a four dimensional Schwarzschild black-hole a test charge near the horizon is repelled by an image charge inside the horizon. In these cases one must subtract the infinite parts to obtain a renormalized Green function or the finite part of self-energy. On a two dimensional static space-time, in contrast to the Maxwell theory in three and four dimensions, as we noticed (after equation (10)) and will discuss later, electrostatic self-energy in QED_2 is a well defined function.

In Schwarzschild coordinate, we consider a non-extremal two dimensional static black hole described by the metric

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2. \quad (25)$$

At the horizon $r = h$, $f(h) = 0$. In this coordinate the equation (7) becomes

$$\left(f(r)\left(\frac{d^2}{dr^2}\right) - \mu^2\right)G(r, r') = f(r)\delta(r, r'). \quad (26)$$

Near the (bifurcate) horizon, i.e. $r \simeq h$, $r > h$, we have $f(r) = \kappa(r - h)$, where 2κ denotes the surface-gravity. Assuming the gauge field tends to zero at infinity and is well behaved at the horizon, the two point function of the gauge field becomes

$$G(r, r') = 2(r_{>} - h)^{\frac{1}{2}}(r_{<} - h)^{\frac{1}{2}}K_1\left(2\mu\sqrt{\frac{r_{>} - h}{\kappa}}\right)I_1\left(2\mu\sqrt{\frac{r_{<} - h}{\kappa}}\right), \quad (27)$$

where $r_{<(>)}$ is the smaller (bigger) of r and r' .

To obtain the effect of the gravitational field on the electrostatic self-interaction, we use the global method used in [12]. If in a free falling coordinates the work δW is needed to displace the charge slowly by a distance δr , then this energy computed at asymptotic infinity, by considering the gravitational red-shift, will be (the space-time is flat at infinity)

$$\delta E = \sqrt{f(r)}\delta W. \quad (28)$$

Using the total mass variation law of Carter [11], and assuming that the metric is unperturbed by the presence of the charges [12], we arrive again to the equation (8). Note that the integral must be taken over T_0^0 from the horizon to infinity. The self-energy of a test charge near the horizon is then

$$E_{self}(x) = e'^2(r - h)K_1\left(2\mu\sqrt{\frac{r - h}{\kappa}}\right)I_1\left(2\mu\sqrt{\frac{r - h}{\kappa}}\right). \quad (29)$$

In contrast to the four dimensional case, there is an attractive force on the test charge near the horizon, which by considering the asymptotic behavior of Bessel functions is independent of vacuum polarization: this attractive force may be related to an image charge inside the horizon, when these charges are near together, that is near the horizon, the effect of vacuum polarization may be disregarded (self-energy is independent of μ). At the horizon the self-energy is zero, and in contrast to the previous example ultraviolet behavior of the Green function doesn't lead to charge confinement. Vanishing of self-energy may be understood as follows: Instead of the black-hole horizon one can consider an image charge inside the black-hole. Then the self-energy of the test charge is

$$E = \frac{-e'^2}{2}(\tilde{G}(r, r) - \tilde{G}(r, r')), \quad (30)$$

where r' is the location of the image charge $-e'$, and \tilde{G} is the Green's function which does not satisfy the Dirichlet boundary condition. In the limit $r = r' = h$ we obtain $E = 0$.

Besides, if we assume $r \rightarrow \infty$, the interaction energy between the charge and its image becomes zero, and we obtain $E = \frac{e'^2}{4\mu}$, which is the self-energy of a test charge on a flat surface. This can be seen explicitly by setting $f(r \rightarrow \infty) = 1$.

3 Heat kernel expansion of electrostatic self-energy

In this part we study the short distance behavior of Green function $G(x, x')$, using the heat kernel of positive elliptic operator $O := -\frac{d}{dx} \frac{1}{\sqrt{g}} \frac{d}{dx} + \mu^2$. This method can be used for slowly varying metrics. We write the heat kernel in the form

$$h(\tau; x, x') = \sum_{n=0}^{\infty} \frac{\tau^{(n-\frac{1}{2})}}{\sqrt{4\pi}} \exp\left(\frac{-\sigma}{2\tau} - \mu^2\tau\right) a_n(x, x'), \quad (31)$$

which satisfies

$$Oh(\tau; x, x') + \frac{\partial h(\tau; x, x')}{\partial \tau} = 0, \quad (32)$$

where $\sigma = \frac{1}{2} \left| \int_{x'}^x g^{\frac{1}{4}}(y) dy \right|^2$, is one half of the square of geodesic distance between (t, x) and (t, x') and τ is the proper-time parameter. $G(x, x')$ is given by

$$G(x, x') = - \int_0^{\infty} h(\tau; x, x') d\tau, \quad (33)$$

provided

$$h(0; x, x') = \delta(x, x'). \quad (34)$$

Therefore

$$G(x, x') = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{\sigma}{2\mu^2}\right)^{(\frac{n}{2}+\frac{1}{4})} K_{n+\frac{1}{2}}(\mu\sqrt{2\sigma}) a_n(x, x') \quad (35)$$

and for $\sigma = 0$,

$$G(x, x) = -\frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (\mu^2)^{-n-\frac{1}{2}} \Gamma(n + \frac{1}{2}) a_n(x, x), \quad (36)$$

which is regular.

Under a scale transformation parameterized by the positive number λ ,

$$\begin{aligned} \sqrt{g}(x) &\rightarrow \lambda \sqrt{g}(x) \\ R(x) &\rightarrow \frac{1}{\lambda} R(x), \end{aligned} \quad (37)$$

the Green function becomes $G_\lambda(\mu^2) = \lambda G(\mu^2 \lambda)$ or $\frac{1}{\lambda} G_\lambda(\frac{\mu^2}{\lambda}) = G(\mu^2)$. We have written the μ dependence of G explicitly. Using (36), we obtain

$$a_n(\lambda) = \frac{a_n(\lambda = 1)}{\lambda^{n-\frac{1}{2}}}. \quad (38)$$

Hence As a polynomial, a_n consists only of m th power of g (including also its derivatives), where $m = \frac{1}{4} - \frac{n}{2}$. In a_n , the order of derivatives is $2n$. For example as we will see, in $a_1(x, x)$, only the terms $g^{-\frac{3}{4}}(x)g'^2(x)$ and $g^{-\frac{5}{4}}(x)g''(x)$ are present.

In order to obtain heat-kernel coefficients we use the relations

$$[\sigma] = [\sigma'] = 0, [\sigma^{(2)}] = \sqrt{g}(x), [\sigma^{(3)}] = \frac{3g'}{4\sqrt{g}}(x). \quad (39)$$

We have shown $\sigma(x, x)$ by $[\sigma]$ and $'$ denotes the first derivative and (n) the n th derivative with respect to x .

By solving the equation (32) for the Seeley coefficients, we obtain a recursion relation

$$\begin{aligned} -\frac{1}{2}g^{-\frac{3}{2}}(x)g'(x)a'_n(x, x') + g^{-\frac{1}{2}}(x)a_n^{(2)}(x, x') - (n+1)a_{n+1}(x, x') + \\ \frac{1}{8}g'(x)\sigma'(x, x')g^{-\frac{3}{2}}(x)a_{n+1}(x, x') - g^{-\frac{1}{2}}\sigma'(x, x')a'_{n+1}(x, x') = 0. \end{aligned} \quad (40)$$

For $n < 0$, $a_{n<0} = 0$. In order to satisfy (34) we must have $a_0(x, x') = g^{\frac{1}{8}}(x)g^{\frac{1}{8}}(x')$. Taking the diagonal value of (40) yields

$$-\frac{1}{2}g^{-\frac{3}{2}}(x)g'(x)[a'_n] + g^{-\frac{1}{2}}(x)[a_n^{(2)}] - (n+1)[a_{n+1}] = 0. \quad (41)$$

For $n = 0$

$$[a_1] = -\frac{1}{2}g^{-\frac{3}{2}}(x)g'(x)[a'_0] + g^{-\frac{1}{2}}[a_0^{(2)}], \quad (42)$$

hence

$$[a_1] = -\frac{11}{64}g^{-\frac{9}{4}}(x)g'^2(x) + \frac{1}{8}g^{-\frac{5}{4}}(x)g^{(2)}(x). \quad (43)$$

For $[a_2]$ we require the diagonal part of a_1 derivatives: $[a'_1]$ and $[a_1^{(2)}]$. Differentiating (40) with respect to x gives

$$\begin{aligned} & \left(-\frac{1}{2}g^{-\frac{3}{2}}(x)g^{(2)}(x) + \frac{3}{4}g^{-\frac{5}{2}}g'^2(x) \right) [a'_n] - g^{-\frac{3}{2}}(x)g'(x)[a_n^{(2)}] + g^{-\frac{1}{2}}[a_n^{(3)}] - \\ & (n+2)[a'_{n+1}] + \frac{1}{8}g^{-1}(x)g'(x)[a_{n+1}] = 0. \end{aligned} \quad (44)$$

For $n = 0$

$$[a'_1] = -\frac{1}{4}g^{-\frac{9}{4}}(x)g'(x)g^{(2)}(x) + \frac{99}{512}g^{-\frac{13}{4}}g'^3 + \frac{1}{16}g^{-\frac{5}{4}}(x)g^{(3)}(x). \quad (45)$$

Another differentiation of (40) with respect to x in the limit $x \rightarrow x'$ leads to the following equation

$$\begin{aligned} & \left(-\frac{1}{2}g^{-\frac{3}{2}}(x)g^{(3)}(x) + \frac{9}{4}g^{-\frac{5}{2}}(x)g'(x)g^{(2)}(x) - \frac{15}{8}g^{-\frac{7}{2}}(x)g'^3(x) \right) [a'_n] + \\ & \left(-\frac{3}{2}g^{(2)}(x)g^{-\frac{3}{2}}(x) + \frac{9}{4}g'^2(x)g^{-\frac{5}{2}}(x) \right) [a_n^{(2)}] + g^{-\frac{1}{2}}[a_n^{(4)}] - \frac{3}{2}g'(x)g^{-\frac{3}{2}}(x)[a_n^{(3)}] + \\ & \left(\frac{1}{4}g^{(2)}(x)g^{-1}(x) - \frac{9}{32}g'^2(x)g^{-2}(x) \right) [a_{n+1}] - \frac{1}{2}g'(x)g^{-1}(x)[a'_{n+1}] - \\ & (n+3)[a_{n+1}^{(2)}] = 0. \end{aligned} \quad (46)$$

Therefore we find

$$\begin{aligned} [a_1^{(2)}] = & -\frac{23}{96}g^{-\frac{9}{4}}(x)g'(x)g^{(3)}(x) + \frac{1}{24}g^{-\frac{5}{4}}(x)g^{(4)}(x) - \frac{31}{192}g^{-\frac{9}{4}}(x)g^{(2)^2}(x) - \\ & \frac{1947}{4096}g^{-\frac{17}{4}}(x)g'^4(x) + \frac{213}{256}g^{-\frac{13}{4}}(x)g'^2(x)g^{(2)}(x). \end{aligned} \quad (47)$$

Equations (41), (45), (47) yield

$$\begin{aligned} [a_2] = & \frac{245}{512}g^{-\frac{15}{4}}(x)g'^2(x)g^{(2)}(x) - \frac{2343}{8192}g^{-\frac{19}{4}}(x)g'^4(x) - \frac{26}{192}g^{-\frac{11}{4}}(x)g'(x)g^{(3)}(x) - \\ & \frac{31}{384}g^{-\frac{11}{4}}(x)g^{(2)^2}(x) + \frac{1}{48}g^{-\frac{7}{4}}(x)g^{(4)}(x). \end{aligned} \quad (48)$$

One can continue this method to obtain other $[a_n]$. Heat kernel coefficient can be expressed in terms of the scalar curvature $R = \frac{1}{2}g^{-\frac{3}{2}}(x)g^{(2)}(x) - \frac{1}{2}g^{-\frac{5}{2}}(x)g'^2(x)$ and $\kappa(x) = \frac{1}{2}g'(x)g^{-1}(x)$ (at the horizon of a black-hole, 2κ is the surface-gravity).

$$\begin{aligned} [a_1] = & \frac{1}{4}g^{\frac{1}{4}}(x)R(x) - \frac{3}{16}g^{-\frac{1}{4}}(x)\kappa^2(x) \\ [a_2] = & \frac{1}{24}g^{-\frac{1}{4}}(x)R^{(2)}(x) - \frac{1}{8}g^{-\frac{1}{4}}(x)\kappa(x)R'(x) - \frac{1}{32}g^{\frac{1}{4}}(x)R^2(x) + \\ & \frac{23}{192}g^{-\frac{1}{4}}(x)\kappa^2(x)R(x) - \frac{23}{512}g^{-\frac{3}{4}}(x)\kappa^4(x). \end{aligned} \quad (49)$$

Hence the self-energy of the charge e' up to the fourth adiabatic order is

$$\begin{aligned}
E_{self.}(x) = & \frac{e'^2}{4} g^{-\frac{1}{4}}(x) \left[\mu^{-1} g^{\frac{1}{2}}(x) + \frac{1}{8} \mu^{-3} \left(g^{\frac{1}{2}}(x) R(x) - \frac{3}{4} \kappa^2(x) \right) + \right. \\
& \frac{3}{32} \mu^{-5} \left(\frac{1}{3} R^{(2)}(x) - \kappa(x) R'(x) - \frac{1}{4} g^{\frac{1}{2}}(x) R^2(x) + \right. \\
& \left. \left. \frac{23}{24} \kappa^2(x) R(x) - \frac{23}{64} g^{-1}(x) \kappa^4(x) \right) \right]. \tag{50}
\end{aligned}$$

Note heat kernel expansion like the WKB method can not be applied at the horizon h or the turning point, i.e. where $g(h) = 0$. To obtain an expression for the self-energy near the horizon one can expand the metric and follows the steps after the equation (25). However the expansion (50), is consistent with the asymptotic behavior of electrostatic self-energy obtained near the horizon. For example Considering asymptotic behavior of Bessel function for large arguments in equation (29) gives the first term in equation (50). This first term is also the same as the result obtained for the small curvature limit (23). For $\sqrt{g} = 1$, (50) is reduced to the flat case result $E_{self.} = \frac{e'^2}{4}$ [6].

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